Varianta 031

Subiectul I

a) $|\vec{v}|=13$. b) $d(D,\prod)=\frac{\sqrt{30}}{10}$. c) $A\left(\frac{\sqrt{10}}{5},\frac{3\sqrt{10}}{5}\right); B\left(-\frac{\sqrt{10}}{5},-\frac{3\sqrt{10}}{5}\right)$. d) $a=-3$. $e)$ 2.

f) $a=-\frac{1}{2}; b=\frac{\sqrt{3}}{2}$.

Subiectul II

1. a) 8. b) $\frac{2}{7}$. c) $C_3^0+C_3^1+C_3^2+C_3^3=2^3=8$. d) $x=0$. e) $\log_3 4>\log_4 3 \Leftrightarrow$

$\Leftrightarrow \log_3 4>\frac{1}{\log_3 4} \Leftrightarrow (\log_3 4)^2>1 \Leftrightarrow \log_3 4>1$, evidenta.

2. a) $f'(x)=3+2\sin x$. b) $\int_0^{\pi}(3x-2\cos x)dx=\frac{3\pi^2}{2}$..

c) $f'(x)=3+2\sin x>0, \forall x\in R$, deoarece $-1\le \sin x\le 1$, deci f este strict crescatoare pe **R**

d) $\lim\limits_{x\to 1}\frac{f(x)-f(1)}{x-1}=f'(1)=3+2\sin 1$. e) $\int_0^1\frac{x}{(x^2+1)^2}dx=\frac{1}{2}\int_0^2\frac{dt}{t^2}=\frac{1}{2}\left(-\frac{1}{t}\right)\Big|_1^2=\frac{1}{4}$.

Subiectul III

a) $g=x^4+1+2x^2-2x^2=(x^2+1)^2-2x^2=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$.
b) $\det V=(x_2-x_1)(x_3-x_1)(x_4-x_1)(x_3-x_2)(x_4-x_2)(x_4-x_3)$ (Vandermonde) .
c) $(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)=0 \Rightarrow x^2-\sqrt{2}x+1=0 \Rightarrow \Delta_1=2-4<0 \Rightarrow$

$\Rightarrow x_{1,2}=\frac{\sqrt{2}\pm i\sqrt{2}}{2}$ sau $x^2+\sqrt{2}x+1=0 \Rightarrow \Delta_2=-2<0 \Rightarrow x_{3,4}=\frac{-\sqrt{2}\pm i\sqrt{2}}{2}$,

$x_1\ne x_2\ne x_3\ne x_4 \Rightarrow \det V\ne 0 \Rightarrow rangV=4$.

d) Vom demonstra pentru elementele de pe coloana 1: $a+bx_1+cx_1^2+dx_1^3=f(x_1)$.

$-d+ax_1+bx_1^2+cx_1^3=dx_1^4+ax_1+bx_1^2+cx_1^3=x_1(a+bx_1+cx_1^2+dx_1^3)=x_1f(x_1)$

$-c-dx_1+ax_1^2+bx_1^3=cx_1^4+dx_1^5+ax_1^2+bx_1^3=x_1^2(a+bx_1+cx_1^2+dx_1^3)=x_1^2f(x_1)$

$-b+c(-x_1)-dx_1^2+ax_1^3=bx_1^4+cx_1^5+dx_1^4+ax_1^3=x_1^3(a+bx_1+cx_1^2+dx_1^3)=x_1^3f(x_1)$

Am folosit $x_1^4=-1$ deoarece x_1 este radacina a lui g $\Rightarrow g(x_1)=0 \Rightarrow x_1^4+1=0$.

e) $\det(A\cdot V)=f(x_1)f(x_2)f(x_3)f(x_4)\begin{vmatrix}1&1&1&1\\x_1&x_2&x_3&x_4\\x_1^2&x_2^2&x_3^2&x_4^2\\x_1^3&x_2^3&x_3^3&x_4^3\end{vmatrix}=f(x_1)f(x_2)f(x_3)f(x_4)\det V \Rightarrow$

$\Rightarrow \det A=f(x_1)f(x_2)f(x_3)f(x_4)$;

f) Singura descompunere a lui g în **R**[X] este cea de la a), care nu este descompunere in **Q**[X].

g) $\det A=0\Rightarrow f(x_1)f(x_2)f(x_3)f(x_4)=0\Rightarrow x_1$ (de exemplu) rădăcină a lui $f\Rightarrow f\,\vdots\, g$, dar grad f < grad g şi $f\in Q[X]\Rightarrow f=0\Rightarrow a=b=c=d=0$.

Subiectul IV

$a)\ a_{n+1}-a_n=\dfrac{1}{(n+1)!}>0\Rightarrow(a_n)_{n\geq1}$ este strict crescator.

$b)\ b_{n+1}-b_n=a_{n+1}-a_n+\dfrac{1}{(n+1)!(n+1)}-\dfrac{1}{n!n}=\dfrac{-1}{n(n+1)(n+1)!}<0\Rightarrow(b_n)_{n\geq1}$ este strict descrescator.

$c)\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}\left(a_n+\dfrac{1}{n!n}\right)=\lim\limits_{n\to\infty}a_n+\lim\limits_{n\to\infty}\dfrac{1}{n!n}=e.$

d) $(a_n)_{n\geq1}$ strict crescator si $\lim\limits_{n\to\infty}a_n=e\Rightarrow a_n<e,(\forall)n\in\mathbf{N}^*.(b_n)_{n\geq1}$ strict descrescator si $\lim\limits_{n\to\infty}b_n=e\Rightarrow e<b_n,(\forall)n\in\ \mathbf{N}^*$. Din cele doua relatii obtinem $a_{n+1}<e<b_{n+1},(\forall)n\in\mathbf{N}^*$.

e) Din punctul c)$\Rightarrow a_n+\dfrac{1}{(n+1)!}<e<a_n+\dfrac{1}{n!n},(\forall)n\in\ \mathbf{N}^*\ \Rightarrow\dfrac{1}{(n+1)!}<e-a_n<\dfrac{1}{n!n},(\forall)n\in\ \mathbf{N}^*$.

f) Presupunem prin reducere la absurd ca $e\in\ \mathbf{Q}\Rightarrow e=\dfrac{p}{q},\ p\in\mathbf{N},\ q\in\mathbf{Z}^*$. Aplicam relatia de la punctul e) pentru n=q si obtinem $\dfrac{1}{(q+1)!}<\dfrac{p}{q}-a_q<\dfrac{1}{q!q}$. Inmultim inegalitatile cu $q!$ vom avea $0<\dfrac{1}{q+1}<p(q-1)!-a_q q!<\dfrac{1}{q!}\leq1\Rightarrow0<p(q-1)!-a_q q!<1.$ Dar $p(q-1)!-a_q q!\in\mathbf{Z}$ fals. Deci $e\notin\mathbf{Q}$.

g) Fie $c_n=\dfrac{n^k}{n!}\Rightarrow c_{n+1}=\dfrac{(n+1)^k}{(n+1)!};c_n>0\Rightarrow(c_n)_{n\geq1}$ mărginit

$\dfrac{c_{n+1}}{c_n}=\left(\dfrac{n+1}{n}\right)^k\dfrac{1}{n+1}=\left(1+\dfrac{1}{n}\right)^k\dfrac{1}{n+1}<1\Rightarrow c_{n+1}=c_n\left(1+\dfrac{1}{n}\right)^k\dfrac{1}{n+1}$ sirul $(c_n)_{n\geq1}$ este convergent si au limita $l\Rightarrow c_n\to l\Rightarrow c_{n+1}\to l\Rightarrow l=l\cdot0=0\Rightarrow\lim\limits_{n\to\infty}\dfrac{n^k}{n!}=0;$

h) Presupunem că există $f,g\in\mathbf{R}[X]$ astfel încât $a_n=\dfrac{f(n)}{g(n)},(\forall)n\in\mathbf{N}^*$. Atunci $\dfrac{1}{(n+1)!}=a_{n+1}-a_n=\dfrac{f(n+1)}{g(n+1)}-\dfrac{f(n)}{g(n)}=\dfrac{u(n)}{v(n)}$ sau $u(n)=\dfrac{v(n)}{(n+1)!}$. Dar $\lim\limits_{n\to\infty}\dfrac{v(n)}{(n+1)!}=0\Rightarrow\lim\limits_{n\to\infty}u(n)=0\Rightarrow u$ este polinomul nul, fals $\left(\text{am avea }\dfrac{1}{(\text{n}+1)!}=0\right)$.